AIAA JOURNAL
Vol. 46, No. 12, December 2008

Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Characterizing Hydrodynamic Loads in Full Unsteady Flow

Larry Silverberg* and Blaine Levedahl†
North Carolina State University, Raleigh, North Carolina 27695-7910

DOI: 10.2514/1.36626

Introduction

CHARACTERIZING full unsteady flow around a body is fundamentally different from characterizing steady flow. Steady flow is characterized through experimentally obtained hydrodynamic derivatives. Using the hydrodynamic derivatives, hydrodynamic loads are expressed in terms of, for example, a freestream velocity U_∞, an angle α, and an angular rate $\dot{\alpha}$. The corresponding full unsteady problem cannot employ hydrodynamic derivatives. Even worse, initial conditions are required at every point in the flow to predict how a full unsteady flow evolves. In physical practice, this is beyond the current state of the art and will likely be beyond the state of the art for years to come. Therefore, the question arises whether there is another way to characterize full unsteady flow and the corresponding hydrodynamic loads.

It is well known that hydrodynamic loads are influenced greatly by the locations of the stagnation lines and the separation lines of a flow. In steady analysis, the Kutta condition prescribes where a flow separates, which in turn helps predict a hydrodynamic load. In fact, it was experimentally demonstrated in two-dimensional problems [1,2] that different flows that happen to produce the same nodes (separation and stagnation) will produce essentially the same hydrodynamic load, and this property was experimentally shown to apply to unsteady flow as well. This phenomenon appears to result from the mathematical nature of the flow's velocity distribution around a body. The velocity distribution is a smoothly varying function that is determined up to a multiplicative constant by its nodes. This property, referred to in this Note as the Kutta principle, would hold when the flow varies continuously as described, whether steady or unsteady. This Note develops a mathematical argument in favor of the Kutta principle and suggests that the hydrodynamics community should consider the Kutta principle as a strategy for characterizing full unsteady flow.

This Note begins with the development of a nodal theorem for estimating hydrodynamic loads. The nodal theorem provides a set of idealized conditions under which the hydrodynamic load is uniquely determined by the nodes. In practice, these conditions are not met exactly, but can be approximated. The remainder of the Note develops approximation methods. Note that the extrapolation of critical information from the nodes of a distribution, although not practiced in the hydrodynamics community, has been successfully exploited in other communities (e.g., [3]).

General Flow over a Body

At any given instant, the hydrodynamic pressure at a point P on a body depends on the velocity $v(P)$. The hydrodynamic load is determined by the pressure and therefore by $v(P)$. The nodes of $v(P)$ are points or lines depending on the formulation or the physics. In planar problems in which only cross sections of flow are being considered and hydrodynamic loads are being examined on a per-length basis, the nodes are points. The nodes are also points in some three-dimensional attached flows, such as in the case of the flow around a sphere. Otherwise, the nodes are lines.

Nodal Points

We first consider the case of nodal points. The locations $P_1, P_2, \ldots$ on the body for which $v(P) = 0$ are called nodal points. The following point theorem for the Kutta principle uniquely determines the flow from the nodal points.

Theorem 1: Let the speed v on the surface of a body be a linear combination of the basis flows $\{v_n(P)\}_{n=1}^N$. Also, take a reference measurement of speed $v_0 = v(P_0)$, where $v_0 \neq 0$. The nodal points are $P_1, P_2, \ldots, P_{N-1}$. The flow is uniquely determined by v_0 and $P_1, P_2, \ldots, P_{N-1}$.

Proof: Write v as

$$v(P,t) = \sum_{n=1}^{N} v_n(P) q_n(t) \tag{1}$$

where $q_1(t), q_2(t), \ldots, q_n(t)$ are coefficients. Evaluating Eq. (1) at $P_0, P_1, P_2, \ldots, P_{N-1}$,

$$\begin{gathered} v_0 = \sum_{n=1}^{N} v_n(P_0) q_n(t) \\ 0 = \sum_{n=1}^{N} v_n(P_1) q_n(t) \quad \cdots \quad 0 = \sum_{n=1}^{N} v_n(P_{N-1}) q_n(t) \end{gathered} \tag{2}$$

In matrix-vector form,

$$\begin{gathered} v_0 \mathbf{1} = \mathbf{A}\mathbf{q} \qquad \mathbf{1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \mathbf{A} = \begin{bmatrix} v_1(P_0) & v_2(P_0) & \cdots & v_N(P_0) \\ v_1(P_1) & v_2(P_1) & \cdots & v_N(P_1) \\ \vdots & \vdots & & \vdots \\ v_1(P_{N-1}) & v_2(P_{N-1}) & \cdots & v_N(P_{N-1}) \end{bmatrix} \end{gathered} \tag{3}$$

Assuming that $\mathbf{A}$ is full rank,

$$\mathbf{q} = \mathbf{A}^{-1}\mathbf{1} v_0 \tag{4}$$

The coefficients are uniquely determined by the locations of the nodal points up to the multiplicative constant v_0. It follows that $v(P)$

Received 12 January 2008; revision received 19 April 2008; accepted for publication 10 June 2008.

*Professor, Mechanical and Aerospace Engineering Department.
†Ph.D. Candidate, Mechanical and Aerospace Engineering Department.

is uniquely determined by the nodal points up to a multiplicative constant. □

Experimental tests can characterize flow around a body on the basis of Theorem 1. Tests are run for different sets of nodal points. In a given run, one reference speed v_0 is measured. In other flows that have the same nodal points, the hydrodynamic loads scale as a function of v_0. In steady flow, hydrodynamic loads are proportional to v_0^2, in which case they scale by v_0^2 [4].

Nodal Point Approximations

Theorem 1 is an idealization, because a flow is not an exact linear combination of basis flows. Next, we consider the attached steady flow around a cylinder [5]. This example is illustrative of a steady or unsteady flow. The speed is $v = \sqrt{v_x^2 + v_y^2}$, where

$$v_x = 1 + \frac{R^2}{r^2} - \frac{2R^2x^2}{r^4} - k\frac{Ry}{r^2}$$

$$v_y = -\frac{2R^2xy}{r^4} + k\frac{Rx}{r^2}$$

and k prescribes circulation. On the surface,

$$v_x = \frac{1}{R^2}[2R^2 - kyR - 2x^2]$$

and

$$v_y = \frac{1}{R^2}x(kR - 2y)$$

In terms of θ, $v_x = \sin\theta[2\sin\theta - k]$ and $v_y = -\cos\theta(2\sin\theta - k)$. The tangential component of velocity is

$$v_t = v_x \sin\theta - v_y \cos\theta = 2\sin\theta - k$$

and the speed is $v = |2\sin\theta - k|$. Theorem 1 can be applied to the tangential component of velocity (replacing v everywhere with v_t). The functions $\{1, \sin\theta\}$ form a basis from which v_t is uniquely and exactly determined by the nodal points P_1 and P_2.

In unattached flow, additional nodal points appear. The vortices around the cylinder alternate in direction, and so the tangential component of velocity on the surface is an oscillating function of θ. Assume $N-1$ nodal points and consider the basis $\{1, \theta, \theta^2, \ldots, \theta^{N-1}\}$ of $N-1$th order polynomials. The tangential component of velocity is written in factored form as

$$v_t = A\prod_{r=1}^{N-1}(\theta - \theta_r) \tag{5}$$

where

$$A = \frac{v_t(\theta_0)}{\prod_{r=1}^{N-1}(\theta_0 - \theta_r)}$$

where θ_0 is the angle of the reference measurement. It is obvious that Theorem 1 holds in the case of the polynomial set by simply inspecting Eq. (5). Equation (5) also suggests a more general method of approximating v from nodal points and nodal lines.

The factors in Eq. (5) can be viewed as the constraints $c_r(\theta) = \theta - \theta_r = 0\,(r = 1, 2, \ldots, N-1)$ placed on the coordinate θ. In terms of x and y, the constraints are $c_r(x, y)$ and a second approximation of the speed is expressed in the general form:

$$v(x, y) = A\prod_{r=1}^{N-1} c_r(x, y) \tag{6}$$

The constraints $c_r(x, y)$ can represent straight lines that intersect the circle at the nodal points. Denote the nodal points on the circle by $(x_r y_r)$ $(r = 1, 2, \ldots, N-1)$. Then for straight-line constraints,

$$v(x, y) = A\prod_{r=1}^{N-1}(R^2 - x_r x - y_r y) \tag{7}$$

Nodal Lines

In three-dimensional flow, the speed is expressed, like Eq. (6), in the general form:

$$v(x, y, z) = A\prod_{r=1}^{N-1} c_r(x, y, z) \tag{8}$$

The constraints c_r are now surfaces. The surfaces intersect the body at nodal points or cut through the body along curved nodal lines. The speed on the body, when expressed in this form, is uniquely determined by its nodes up to a multiplicative constant, such as in the case of Theorem 1.

Consider attached steady flow around a sphere. The speed is $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$, where

$$v_x = 2 + \frac{R^3}{r^3} - \frac{3R^3x^2}{r^5} - k_z\frac{Ry}{r^2} + k_y\frac{Rz}{r^2}$$
$$v_y = -\frac{3R^3xy}{r^5} + k_z\frac{Rx}{r^2} - k_x\frac{Rz}{r^2}$$
$$v_z = -\frac{3R^3xz}{r^5} + k_x\frac{Ry}{r^2} - k_y\frac{Rx}{r^2}$$

and k_x, k_y, and k_z prescribe circulation about the x, y, and z axes, respectively. On the surface,

$$v_x = \frac{1}{R^2}[3R^2 - 3x^2 + R(-k_z y + k_y z)]$$
$$v_y = \frac{1}{R^2}[-3xy + R(k_z x - k_x z)]$$
$$v_z = \frac{1}{R^2}[-3xz + R(k_x y - k_y x)]$$

The θ and ϕ components of velocity are

$$v_\theta = -v_x\sin\theta + v_y\cos\theta = -3\sin\theta - k_x\cos\phi\cos\theta - k_y\cos\phi\sin\theta + k_z\sin\phi$$
$$v_\phi = v_x\cos\phi\cos\theta + v_y\cos\phi\sin\theta - v_z\sin\phi = 3\cos\phi\cos\theta - k_x\sin\theta + k_y\cos\theta$$

The speed is $v = \sqrt{v_\theta^2 + v_\phi^2}$. The constraints $c_r(x, y, z)$ can represent flat planes that intersect the sphere at the nodal points x_r, y_r, and z_r $(r = 1, 2, \ldots, N-1)$. For flat-plane constraints that intersect at nodal points,

$$v(x, y, z) = A\prod_{r=1}^{N-1}(R^2 - x_r x - y_r y - z_r z) \tag{9}$$

Illustrative Examples

Consider first the cylinder and then the sphere. Figure 1 shows the flow around a cylinder for $k = -1.75, -1.00, 0, 1.00,$ and 1.75. The flow was measured at the top of the cylinder ($\theta_0 = 90$ deg). The middle column compares the exact $v_t(\theta)$ with the polynomial approximation of Eq. (5). For improved accuracy, an eighth-order polynomial was used that has zeros at $\theta_1 + j360$ deg and $\theta_2 - j360$ deg ($j = -1, 0, 1,$ and 2). The right column compares the exact $v(\theta)$ with the constrained method of Eq. (7).

Figure 2 shows the flow around a sphere for $k_z = -1.75, -1.00, 0, 1.00,$ and 1.75. The flow was measured at the top of the sphere. The right column compares the exact $v(\theta, \phi)$ with the constrained approach of Eq. (9).

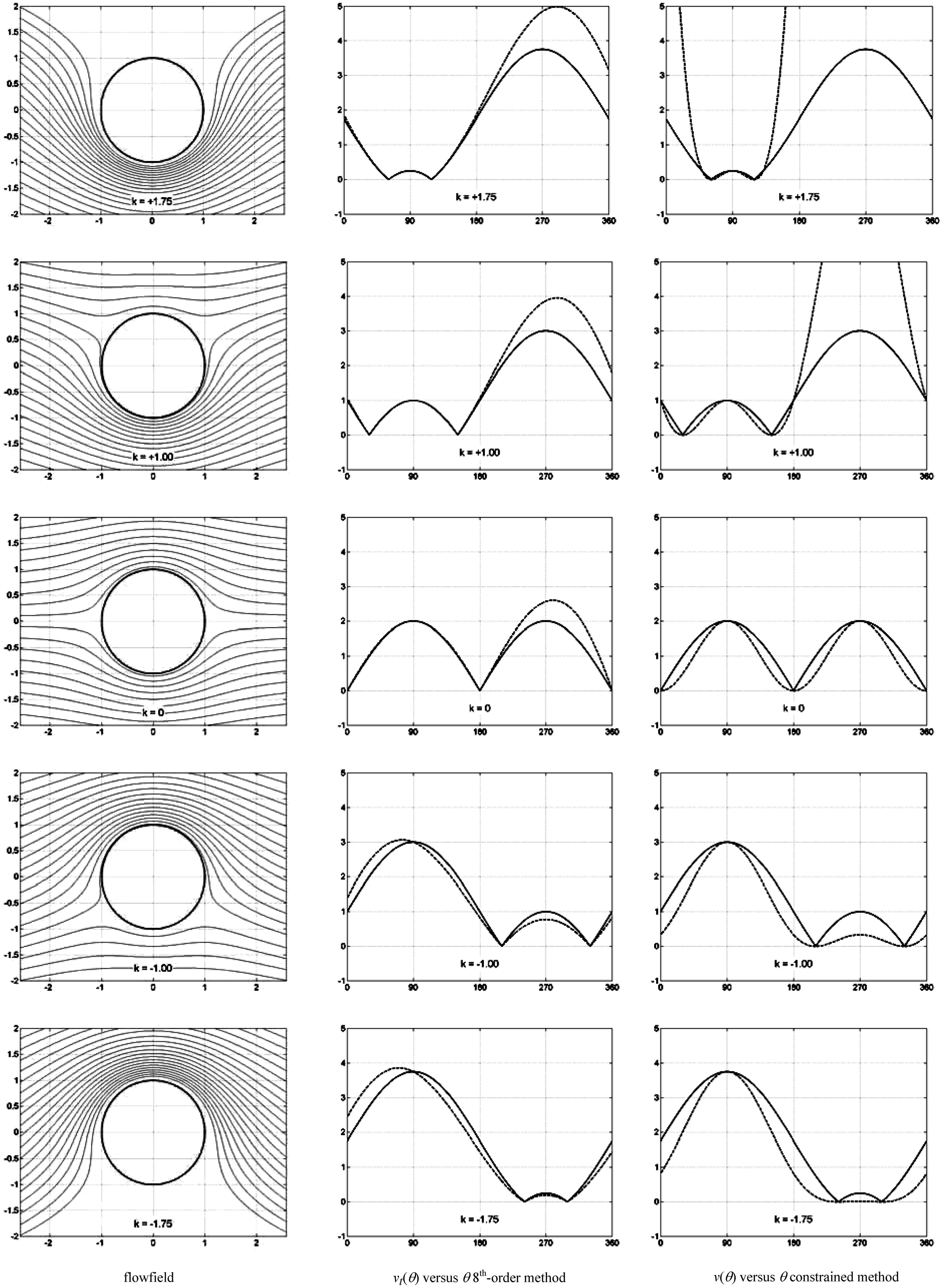

Fig. 1 Approximating $v(\theta)$ over a cylinder: exact (solid) and approximate (dashed).

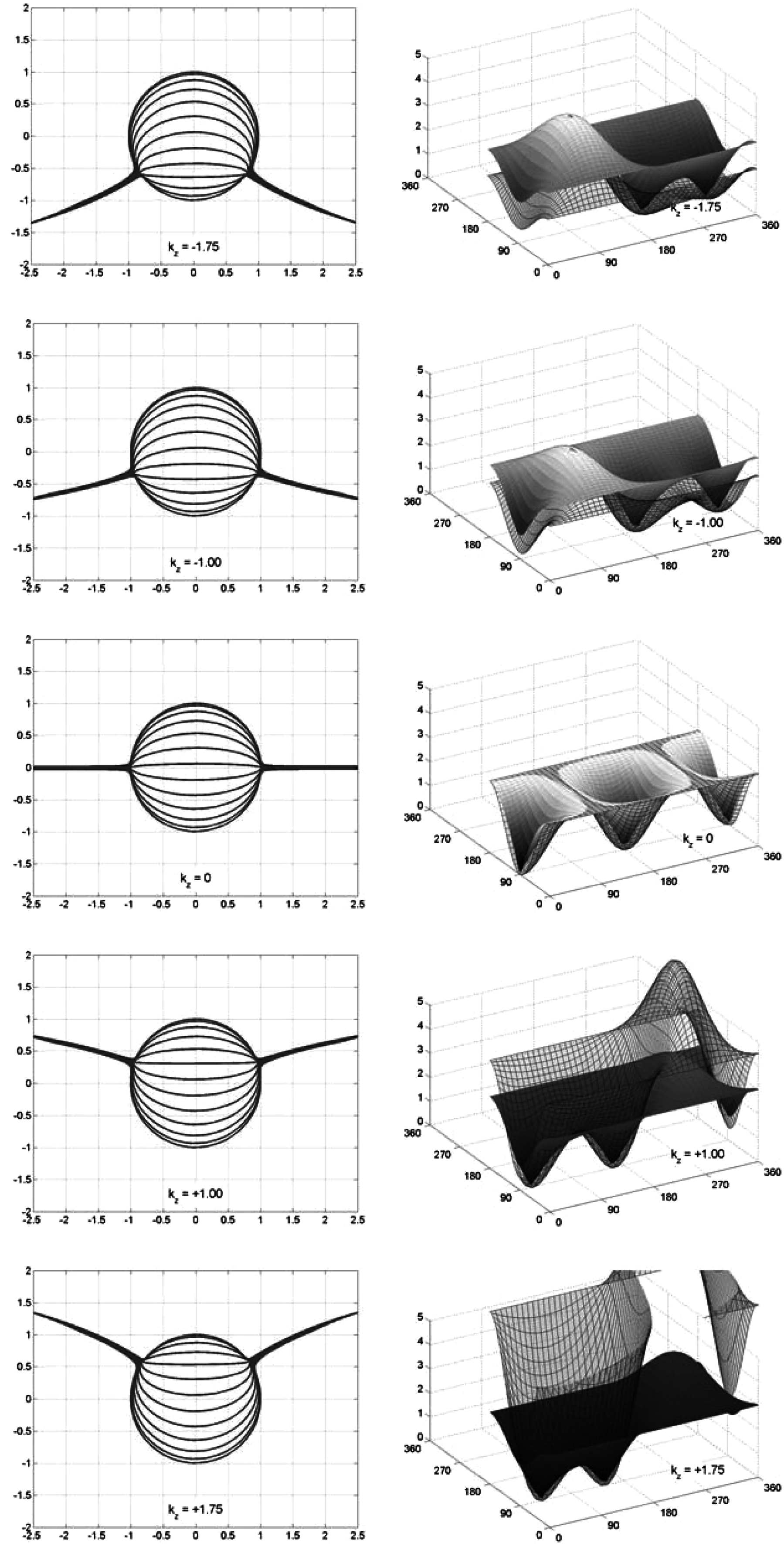

Fig. 2 Approximating $v(\theta)$ over a sphere: exact (solid) and approximate (grid).

Referring to Fig. 1, the eighth-order approximation shown in the middle column is more accurate than the constrained method shown in the right column. The errors in both approximations are larger when the nodal points in the flow are close to the measurement point. It appears to be best to measure the flow away from the nodal points. Figure 2 shows similar results. The approximation is best when the measurement is away from the nodal points.

Conclusions

This Note developed a mathematical argument in favor of the Kutta principle. The Note then showed two methods of approximating the velocity distribution of a flow that has been specified by its nodes. The method contained in this Note is a potential strategy for characterizing full unsteady flow. The Kutta principle could be applied in testing or even in onboard control problems. For example, using the Kutta principle, a method of sensing the locations of nodal lines could be developed to obtain a real-time estimate of the hydrodynamic load. The estimate could be integrated into the control system for more efficient navigation or to prevent vehicle instability. Such an estimate would also make it possible to develop stable vehicles that can maneuver rapidly.

Acknowledgment

This work was supported in part by the U.S. Office of Naval Research grant N00014-06-1-0593.

References

[1] Goman, M., and Khrabrov, A., "State-Space Representation of Aerodynamic Characteristics of an Aircraft at High Angles of Attack," *Journal of Aircraft*, Vol. 31, No. 5, Sept.–Oct. 1994, pp. 1109–1115.

[2] Mangalam, S. M., "Real-Time Extraction of Hydrodynamic Flow Characteristics Using Surface Signatures," *IEEE Journal of Oceanic Engineering*, Vol. 29, No. 3, July 2004, pp. 622–630.

[3] Weaver, L., Jr., and Silverberg, L., "Node Control of Uniform Beams Subject to Various Boundary Conditions," *Journal of Applied Mechanics*, Vol. 59, No. 4, Dec. 1992, pp. 983–990. doi:10.1115/1.2894070

[4] Newman, J. N., *Marine Hydrodynamics*, MIT Press, Cambridge, MA, 1977.

[5] Anderson, J. D., *Fundamentals of Aerodynamics*, McGraw–Hill, New York, 1991.

T. Jackson
Associate Editor